

UNCONDITIONALLY p -CONVERGING OPERATORS AND DUNFORD-PETTIS PROPERTY OF ORDER p

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ABSTRACT. In the present paper we study unconditionally p -converging operators and Dunford-Pettis property of order p . New characterizations of unconditionally p -converging operators and Dunford-Pettis property of order p are established. Six quantities are defined to measure how far an operator is from being unconditionally p -converging. We prove quantitative versions of relationships of completely continuous operators, unconditionally p -converging operators and unconditionally converging operators. We further investigate possible quantifications of the Dunford-Pettis property of order p .

1. INTRODUCTION AND NOTATIONS

Throughout the paper, p^* denotes the conjugate number of p for $1 \leq p < \infty$; if $p = 1$, l_{p^*} plays the role of c_0 . X, Y will denote real (or complex) Banach spaces and $\mathcal{L}(X, Y)$ the space of all the operators (=continuous linear maps) between X and Y . $\mathcal{K}(X, Y)$ denotes the space of all the compact operators between X and Y . Let X be a Banach space, $1 \leq p < \infty$ and we denote $l_p(X)$ by the set of all p -summable sequences in X with the natural norm $\|(x_n)_n\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{\frac{1}{p}}$. Let $l_p^w(X)$ be the set of all weakly p -summable sequences in X . Then $l_p^w(X)$ is a Banach space with the norm

$$\|(x_n)_n\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x^*, x_n \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}, \quad \forall (x_n)_n \in l_p^w(X).$$

It is a well-known result of A. Grothendieck ([15], [12, Proposition 2.2]) that the canonical correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{L}(l_{p^*}, X)$ onto $l_p^w(X)$. A sequence $(x_n)_n \in l_p^w(X)$ is *unconditionally p -summable* if

$$\sup\left\{\left(\sum_{n=m}^{\infty} |\langle x^*, x_n \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

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We denote the set of all unconditionally p -summable sequences on X by $l_p^u(X)$. It is obvious that $(x_n)_n$ is unconditionally 1-summable if and only if $(x_n)_n$ is unconditionally summable. J. H. Fourie and J. Swart proved that the same correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{K}(l_{p^*}, X)$ onto $l_p^u(X)$ (see [14]). Let us recall that an operator $T : X \rightarrow Y$ is *unconditionally converging* if T takes weakly 1-summable sequences to unconditionally 1-summable sequences. For $p = \infty$, the space $l_\infty^u(X)$ is identical to $c_0(X)$, the space of all norm null sequences in X . Henceforth, for $p = \infty$, we refer to consider the space $c_0^w(X)$ of weakly null sequences in X , instead of $l_\infty^w(X) = l_\infty(X)$. Recall that an operator $T : X \rightarrow Y$ is *completely continuous* if T takes weakly null sequences to norm null sequences. It is well-known that p -summing operators are precisely those operators which take weakly p -summable sequences (unconditionally p -summable sequences) to p -summable sequences. A natural question arises: what are operators which take weakly p -summable sequences to unconditionally p -summable sequences? This is the starting point of our investigation. The paper is organized as follows:

In Section 2, we introduce the concept of unconditionally p -converging operators ($1 \leq p \leq \infty$), which is the extension of unconditionally converging operators and completely continuous operators. It is proved that unconditionally p -converging operators coincide with the p -converging operators introduced by J. M. F. Castillo and F. Sánchez in [7] although their original definitions are different. New concepts of weakly p -Cauchy sequences and weakly p -limited sets are introduced to characterize unconditionally p -converging operators. We establish characterizations of weakly p -limited sets and investigate connections between weakly p -limited sets and relatively norm compact sets. A counterexample is constructed to show that an operator is unconditionally p -converging not precisely when its second adjoint is.

Section 3 is concerned with Dunford-Pettis property of order p (DPP_p for short) introduced in [7], which is a generalization of the classical Dunford-Pettis property. It turns out that many classical spaces failing Dunford-Pettis property enjoy DPP_p , such as Hardy space H^1 and Lorentz function spaces $\Lambda(W, 1)$. In this section, we use weakly p -Cauchy sequences and weakly p -limited sets to characterize DPP_p . New characterizations of DPP_p in dual spaces are obtained. We also introduce the notion of hereditary Dunford-Pettis property of order p and establish its characterizations. In particular, we prove that a Banach space X has the hereditary DPP_p if and only if every weakly p -summable sequence in X admits a weakly 1-summable subsequence.

Finally, the surjective Dunford-Pettis property of order p , a formally weaker property than DPP_p , is introduced and its characterizations are obtained.

In the last two sections of the present paper we investigate possibilities of quantifying unconditionally p -converging operators and the Dunford-Pettis property of order p . This is inspired by a large number of recent results on quantitative versions of various theorems and properties of Banach spaces (see [1,3,13,17,18,19]). Section 4 contains quantitative versions of the implications among three classes of operators—completely continuous, unconditionally p -converging and unconditionally converging ones. M. Kačena, O. F. K. Kalenda and J. Spurný have already defined a quantity measuring how far an operator is from being completely continuous in [17]. In this section, we define another equivalent quantity measuring complete continuity of an operator. We further define six quantities measuring how far an operator is from being unconditionally p -converging. Moreover, we show that one of the six new quantities is equal to the quantity defined in [20] to measure how far an operator is unconditionally converging in case of $p = 1$.

In Section 5 we introduce a new locally convex topology and give two topological characterizations of Dunford-Pettis property of order p . Using the introduced quantity measuring unconditional p -convergence of an operator and the new locally convex topology, we show that the Dunford-Pettis property of order p is automatically quantitative in a sense. We also define two quantities measuring how far a set is weakly p -limited. One of the two new quantities is used to quantify the Dunford-Pettis property of order p . The other is used to define a stronger quantitative version of Dunford-Pettis property of order p . Several characterizations of this quantitative version of Dunford-Pettis property of order p are established.

The reader is referred to [12] and [22] for any unexplained notation or terminology.

2. UNCONDITIONALLY p -CONVERGING OPERATORS

Definition 2.1. Let $1 \leq p \leq \infty$. We say that an operator $T : X \rightarrow Y$ is *unconditionally p -converging* if T takes a weakly p -summable sequence $(x_n)_n \in l_p^w(X)$ ($(x_n)_n \in c_0^w(X)$ for $p = \infty$) to an unconditionally p -summable sequence $(Tx_n)_n \in l_p^u(Y)$ ($(x_n)_n \in c_0(Y)$ for $p = \infty$).

We begin with a simple, but extremely useful, characterization of unconditionally p -converging operators.

Theorem 2.1. *Let $1 \leq p < \infty$. The following are equivalent for an operator $T : X \rightarrow Y$:*

- (1) *T is unconditionally p -converging;*
- (2) *TS is compact for any operator $S \in \mathcal{L}(l_{p^*}, X)(\mathcal{L}(c_0, X)$ for $p = 1$).*

Proof. (1) \Rightarrow (2). Let $S \in \mathcal{L}(l_{p^*}, X)(1 < p < \infty)(\mathcal{L}(c_0, X)$ for $p = 1$). By the ideal property of unconditionally p -converging operators, TS is unconditionally p -converging. Since $(e_n)_n$ is weakly p -summable in $l_{p^*}(1 < p < \infty)(c_0$ for $p = 1$), $(TSe_n)_n$ is unconditionally p -summable. Then there exists a compact operator $R : l_{p^*} \rightarrow X$ such that $Re_n = TSe_n(n = 1, 2, \dots)$. Thus TS is compact.

(2) \Rightarrow (1). Let $(x_n)_n \in l_p^w(X)$. Then there exists an operator $S : l_{p^*} \rightarrow X(1 < p < \infty)(S : c_0 \rightarrow X$ for $p = 1)$ such that $Se_n = x_n(n = 1, 2, \dots)$. By (2), we get $(TSe_n)_n$ is unconditionally p -summable. Thus TS is unconditionally p -converging. □

Before another frequently useful characterization of unconditionally p -converging operators is given, we recall the notion of weakly p -convergent sequences introduced in [8]. A sequence $(x_n)_n$ in a Banach space X is said to be weakly p -convergent to $x \in X(1 \leq p \leq \infty)$ if the sequence $(x_n - x)_n$ is weakly p -summable in X . Weakly ∞ -convergent sequences are simply the weakly convergent sequences. It is natural to generalize weakly Cauchy sequences to the general case $1 \leq p \leq \infty$.

Definition 2.2. Let $1 \leq p \leq \infty$. We say that a sequence $(x_n)_n$ in a Banach space X is *weakly p -Cauchy* if for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(x_{k_n} - x_{j_n})_n$ is weakly p -summable in X .

Obviously, every weakly p -convergent sequence is weakly p -Cauchy, and the weakly ∞ -Cauchy sequences are precisely the weakly Cauchy sequences.

Theorem 2.2. *Let $1 \leq p \leq \infty$. The following statements about an operator $T : X \rightarrow Y$ are equivalent:*

- (1) *T is unconditionally p -converging;*
- (2) *T sends weakly p -convergent sequences onto norm convergent sequences;*
- (3) *T sends weakly p -Cauchy sequences onto norm convergent sequences.*

Proof. (1) \Rightarrow (2). Suppose that $(x_n)_n$ is weakly p -convergent in X . We may assume that $(x_n)_n$ is weakly p -summable. Then there exists an operator $S : l_{p^*} \rightarrow X, 1 < p < \infty(S : c_0 \rightarrow X$ for $p = 1)$ such that $Se_n = x_n(n = 1, 2, \dots)$. By Theorem 2.1, TS is

compact and hence $(TSe_n)_n$ is relatively compact. Consequently, $\lim_{n \rightarrow \infty} \|TSe_n\| = 0$.

(2) \Rightarrow (3). Let $(x_n)_n$ be a weakly p -Cauchy sequence in X . By (2), for each pair of strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers, the sequence $(Tx_{k_n} - Tx_{j_n})_n$ converges to 0 in norm and hence $(Tx_n)_n$ converges in norm.

(3) \Rightarrow (1). Suppose that T is not unconditionally p -converging. By Theorem 2.1, the operator TS is non-compact for some operator $S \in \mathcal{L}(l_{p^*}, X)$ ($1 < p < \infty$) ($\mathcal{L}(c_0, X)$ for $p = 1$). Then there exists a weakly null sequence $(z_n)_n$ in l_{p^*} ($1 < p < \infty$) (c_0 for $p = 1$) such that $\|TSz_n\| > \epsilon_0 > 0$ ($n = 1, 2, \dots$). By passing to subsequences, we may assume that the sequence $(z_n)_n$ is equivalent to the unit vector basis $(e_n)_n$ in l_{p^*} . Let $R : l_{p^*} \rightarrow l_{p^*}$ be an isomorphic embedding with $Re_n = z_n$ ($n = 1, 2, \dots$). Let $x_n = SRe_n$. Then $(x_n)_n$ is weakly p -summable in X and hence weakly p -Cauchy. By the assumption, $(Tx_n)_n$ converges to 0 in norm, but $\|Tx_n\| > \epsilon_0 > 0$ ($n = 1, 2, \dots$), which is a contradiction. □

It should be noted that Theorem 2.2(2) is the definition of the so called p -converging operators defined by J. M. F. Castillo and F. Sánchez in [7]. In this note, we use the terminology unconditionally p -converging operators instead of p -converging operators.

Recall that a subset K of a Banach space X is *relatively weakly p -compact* ($1 \leq p < \infty$) if K is contained in $S(B_{l_{p^*}})$ for $1 < p < \infty$ ($S(B_{c_0})$ for $p = 1$) for some operator S from l_{p^*} (c_0 for $p = 1$) into X (see [25]). A subset K of a Banach space X is said to be *relatively weakly p -precompact* if every sequence in K admits a weakly p -convergent subsequence (see [6]). Bessaga-Pełczyński Selection Principle yields that every relatively weakly p -compact set is relatively weakly p -precompact for any $1 < p < \infty$. But the converse needs not to be true. Let $X = (\sum_{n=1}^{\infty} l_1^n)_{p^*}$ ($1 < p < \infty$). It follows from Bessaga-Pełczyński Selection Principle that B_X is relatively weakly p -precompact. But B_X is not relatively weakly p -compact because X is not isomorphic to a quotient of l_{p^*} . Another counterexample is L_p ($1 < p < \infty, p \neq 2$). For each $1 < p < \infty, p \neq 2$, B_{L_p} is relatively weakly r -precompact, where $r = \max(p^*, 2)$, but is not relatively weakly r -compact because such L_p is not isomorphic to a quotient of l_{r^*} .

By using the weakly p -Cauchy sequences, we can correspondingly define the conditionally weakly p -compact sets as follows:

Definition 2.3. Let $1 \leq p \leq \infty$. We say that a subset K of a Banach space X is *conditionally weakly p -compact* if every sequence in K admits a weakly p -Cauchy subsequence.

The following result, which follows from Theorem 2.2, says that unconditionally p -converging operators are precisely those operators that send conditionally weakly p -compact subsets onto relatively norm compact subsets.

Theorem 2.3. Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p < \infty$. The following statements are equivalent:

- (1) T is unconditionally p -converging;
- (2) T maps relatively weakly p -precompact subsets onto relatively norm compact subsets;
- (3) T maps conditionally weakly p -compact subsets onto relatively norm compact subsets;
- (4) T maps relatively weakly p -compact subsets onto relatively norm compact subsets.

Definition 2.4. Let X be a Banach space and $1 \leq p < \infty$. We say that a bounded subset K of X^* is *weakly p -limited* if $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$ for every $(x_n)_n \in l_p^w(X)$.

The following result, an immediate consequence of Theorem 2.2, is a characterization of unconditionally p -converging operators in terms of weakly p -limited subsets.

Theorem 2.4. Let $1 \leq p < \infty$. The following are equivalent for an operator $T : X \rightarrow Y$:

- (1) T is unconditionally p -converging;
- (2) T^* maps bounded subsets of Y^* onto weakly p -limited subsets of X^* .

J.M.F.Castillo and F.Sánchez said that a Banach space $X \in W_p$ ($1 \leq p < \infty$) if any bounded sequence in X admits a weakly p -convergent subsequence (see [8]). We use this notion to characterize weakly p -limited sets.

Theorem 2.5. Let $1 < p < \infty$ and X be a Banach space. The following statements are equivalent about a bounded subset K of X^* :

- (1) K is weakly p -limited;
- (2) For all spaces $Y \in W_p$ and for every operator T from Y into X , the subset $T^*(K)$ is relatively norm compact;

(3) For every operator T from l_{p^*} into X , the subset $T^*(K)$ is relatively norm compact.

Proof. (1) \Rightarrow (2). Let T be an operator from $Y \in W_p$ into X such that $T^*(K)$ is not relatively norm compact. Then there exists a sequence $(x_n^*)_n$ in K such that $(T^*x_n^*)_n$ admits no norm convergent subsequences. Since Y^* is reflexive, by passing to a subsequence if necessary we may assume that $(T^*x_n^*)_n$ converges weakly to some $y^* \in Y^*$ and $\|T^*x_n^* - y^*\| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $n \in \mathbb{N}$. For each n , choose y_n with $\|y_n\| \leq 1$ such that $|\langle T^*x_n^* - y^*, y_n \rangle| > \epsilon_0$. Since $Y \in W_p$, by passing to a subsequence again if necessary one can assume that the sequence $(y_n)_n$ is weakly p -convergent to some $y \in Y$. Thus, by hypothesis, we get $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, Ty_n - Ty \rangle| = 0$. Note that, for each $n \in \mathbb{N}$,

$$|\langle T^*x_n^* - y^*, y_n \rangle| \leq |\langle x_n^*, Ty_n - Ty \rangle| + |\langle x_n^*, Ty \rangle - \langle y^*, y \rangle| + |\langle y^*, y - y_n \rangle|.$$

This implies that $\lim_{n \rightarrow \infty} \langle T^*x_n^* - y^*, y_n \rangle = 0$, which is a contradiction.

(2) \Rightarrow (3) is immediate because $l_{p^*} \in W_p$;

(3) \Rightarrow (1). Let $(x_n)_n \in l_p^w(X)$. Then there exists an operator T from l_{p^*} into X such that $Te_n = x_n$ for all $n \in \mathbb{N}$. It follows from (3) that $T^*(K)$ is relatively norm compact. By the well-known characterization of relatively norm compact subsets of l_p , one can derive that $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$. □

By Theorem 2.5, we see that relatively norm compact sets are weakly p -limited. But Theorem 2.4 demonstrates that there are many weakly p -limited sets which are not relatively norm compact. Indeed, for each $1 < p < \infty$ and for each $1 < r < p^*$, the identity map I_r on l_r is unconditionally p -converging and hence the unit ball $B_{l_r^*}$ of l_r^* is weakly p -limited. In the following result, we use biorthogonal sequences to characterize weakly p -limited sets which are not relatively norm compact.

Theorem 2.6. *Suppose that X is reflexive and K is a weakly p -limited subset of X^* . If K is not relatively norm compact, then there exists a seminormalized biorthogonal sequence $(x_n, x_n^*)_n$ in $X \times (K - K)$ such that $(x_n^*)_n$ is a basic sequence and $(x_n)_n$ has no weakly p -Cauchy subsequence.*

Proof. Suppose that K is not relatively norm compact, and let $(f_n)_n$ be a sequence in K with no norm convergent subsequence. Since X is reflexive, we may assume that the sequence $(f_n)_n$ converges weakly. Then there exist two strictly increasing sequences $(k_n)_n$ and $(j_n)_n$ of positive integers and $\epsilon_0 > 0$ such that $\|f_{k_n} - f_{j_n}\| > \epsilon_0$

for all $n \in \mathbb{N}$. Let $x_n^* = f_{k_n} - f_{j_n} \in (K - K)$. Then $(x_n^*)_n$ is weakly null. By Bessaga-Pełczyński Selection Principle, we can assume that $(x_n^*)_n$ is a basic sequence. Let $(x_n^{**})_n$ be the associated sequence of coefficient functionals, and for each $n \in \mathbb{N}$, let $x_n \in X$ be a Hahn-Banach extension of x_n^{**} to all of X^* . Then the sequence $(x_n, x_n^*)_n$ is seminormalized and biorthogonal.

It remains to show that $(x_n)_n$ has no weakly p -Cauchy subsequence. If $(y_n)_n$ is a weakly p -Cauchy subsequence of $(x_n)_n$, then $(y_{n+1} - y_n)_n$ is weakly p -summable. Since K is weakly p -limited, the subset $K - K$ is also weakly p -limited, which implies that $\lim_{n \rightarrow \infty} \sup_k | \langle x_k^*, y_{n+1} - y_n \rangle | = 0$. This is impossible because $(x_n, x_n^*)_n$ is biorthogonal. □

A consequence of Theorem 2.6 is that for any $1 < p < \infty$, there exists a relatively weakly compact sequence that admits no weakly p -Cauchy subsequence. Moreover, it should be noted that the converse of Theorem 2.6 is true. Actually, it is easy to verify that if K is a subset of X^* and the sequence $(x_n, x_n^*)_n$ in $X \times (K - K)$ is biorthogonal with $\sup_n \|x_n\| < \infty$, then K is not relatively norm compact.

The following result shows that an operator is unconditionally p -converging not precisely when its second adjoint is.

Theorem 2.7.

- (1) Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p \leq \infty$. If T^{**} is unconditionally p -converging, then T is unconditionally p -converging;
- (2) For each $1 \leq p \leq \infty$, there exists an unconditionally p -converging operator T , but T^{**} is not unconditionally p -converging.

Proof. (1). By the ideal property of unconditionally p -converging operators, $J_Y T$ is unconditionally p -converging, where $J_Y : Y \rightarrow Y^{**}$ is the canonical mapping. Let $S \in \mathcal{L}(l_p^*, X)$ ($1 < p < \infty$) ($\mathcal{L}(c_0, X)$ for $p = 1$). By Theorem 2.1, $J_Y T S$ is compact and hence $T S$ is compact. Again by Theorem 2.1, T is unconditionally p -converging.

(2). J. Bourgain and F. Delbaen (see [5]) constructed a Banach space X_{BD} such that X_{BD} has the Schur property and X_{BD}^{**} is isomorphically universal for separable Banach spaces. Since X_{BD} has the Schur property, every operator from l_p ($1 < p < \infty$) and from c_0 into X_{BD} is compact. By Theorem 2.1, every operator with domain X_{BD} is unconditionally p -converging for each $1 \leq p < \infty$. In particular, the identity map $I_{X_{BD}}$ on X_{BD} is unconditionally p -converging. But since X_{BD}^{**} is isomorphically

universal for separable Banach spaces, there exists a closed subspace X_{p^*} (X_0 for $p = 1$) of X_{BD}^{**} such that X_{p^*} is isomorphic to l_{p^*} for $1 < p < \infty$ (X_0 is isomorphic to c_0 for $p = 1$). This implies that $I_{X_{BD}}^{**} = I_{X_{BD}^{**}}$ is not l_{p^*} -strictly singular for $1 < p < \infty$ (c_0 -strictly singular for $p = 1$). Thus $I_{X_{BD}}^{**} = I_{X_{BD}^{**}}$ is not unconditionally p -converging. For $p = \infty$, the identity map $I_{X_{BD}}$ is obviously completely continuous, but $I_{X_{BD}}^{**} = I_{X_{BD}^{**}}$ is not completely continuous because X_{BD}^{**} has not the Schur property. \square

3. DUNFORD-PETTIS PROPERTY OF ORDER p

Let us recall that a Banach space X has the *Dunford-Pettis property* (in short, DPP) if for every Banach space Y , every weakly compact operator $T : X \rightarrow Y$ is completely continuous (see [16]). An operator $T : X \rightarrow Y$ is said to be *weakly compact* if TB_X is relatively weakly compact in Y . J. M. F. Castillo and F. Sánchez extended the classical Dunford-Pettis property to the general case for $1 \leq p \leq \infty$ in [7]. Let $1 \leq p \leq \infty$. A Banach space X is said to have the *Dunford-Pettis property of p* (in short, DPP_p) if for every Banach space Y , every weakly compact operator $T : X \rightarrow Y$ is unconditionally p -converging. Many classical spaces failing the DPP enjoy the DPP_p . A simple observation is that if a Banach space X has cotype $q < \infty$, then X has the DPP_p for any $1 < p < q^*$. Thus, the classical Hardy space H^1 , which fails the DPP (see [10]), has the DPP_p for any $1 < p < 2$. It is known that all the Lorentz function spaces $\Lambda(W, 1)$'s fail the DPP (see [10]). But there are certain positive results for DPP_p . For example, if we take $W(t) = \frac{1}{2\sqrt{t}}$, $t \in (0, 1]$, then the space $\Lambda(W, 1)$ has the DPP_p for some $1 < p \leq 2$. Another non-reflexive space failing the DPP is the interesting space L built in [21]. Indeed, it was shown in [4] that even duals of L fail the DPP and odd duals of L fail the surjective DPP, which is genuinely weaker than the DPP. Moreover, F. Bombal, P. Cembranos and J. Mendoza proved that for any $1 \leq p < \infty$, every operator from L into l_p is compact (see [4]). This means that L^* has the DPP_p for any $1 < p < \infty$. More examples can be found in [7].

Let us start with a characterization of the DPP_p by means of weakly p -limited sets.

Theorem 3.1. *Let $1 < p < \infty$. A Banach space X has the DPP_p if and only if each relatively weakly compact subset of X^* is weakly p -limited.*

Proof. The sufficient part follows immediately from Theorem 2.4. On the other hand, let K be a relatively weakly compact subset of X^* . By the Davis-Figiel-Johnson-Pełczyński factorization lemma (see [9]), there exists a reflexive space Z , which is a

linear subspace of X^* , such that the inclusion map $J : Z \rightarrow X^*$ is bounded and the unit ball B_Z of Z contains K . Since Z is reflexive, there is an operator $T : X \rightarrow Z^*$ such that $T^* = J$. By the assumption, T is unconditionally p -converging. By Theorem 2.4, the set $T^*(B_Z) = J(B_Z) = B_Z$ is weakly p -limited in X^* . Thus K is also weakly p -limited. \square

Let us remark that for each $1 < p < \infty$, there exists a weakly p -limited set which is not relatively weakly compact. Indeed, we take $X = L^*$, where the space L is built in [21]. As mentioned above, the identity I_X on X is unconditionally p -converging for each $1 < p < \infty$. It follows from Theorem 2.4 that the unit ball B_{X^*} is weakly p -limited, but it is not weakly compact because the space L is non-reflexive.

The following result is an internal characterization of the DPP_p . It is a refinement of [7, Proposition 3.2].

Theorem 3.2. *Let $1 < p < \infty$ and X be a Banach space. The following are equivalent:*

- (1) X has the DPP_p ;
- (2) Every weakly compact operator T from X into c_0 is unconditionally p -converging;
- (3) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every weakly p -Cauchy sequence $(x_n)_n$ in X and every weakly null sequence $(x_n^*)_n$ in X^* ;
- (4) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n)_n \in l_p^w(X)$ and every weakly null sequence $(x_n^*)_n$ in X^* ;
- (5) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n)_n \in l_p^w(X)$ and every weakly Cauchy sequence $(x_n^*)_n$ in X^* .

Proof. (1) \Rightarrow (2) is trivial. (2) \Rightarrow (3). Given a weakly p -Cauchy sequence $(x_n)_n$ in X and a weakly null sequence $(x_n^*)_n$ in X^* . Define an operator $T : X \rightarrow c_0$ by $Tx = (\langle x_n^*, x \rangle)_n$. Since $(x_n^*)_n$ converges to 0 weakly, T^* is weakly compact and so is T . By (2), T is unconditionally p -converging. By Theorem 2.2, $(Tx_n)_n$ converges to some $\xi = (\xi_k)_k \in c_0$ in norm. Let $\epsilon > 0$. There exists a positive integer N_1 such that $\|Tx_n - \xi\| < \frac{\epsilon}{2}$ for all $n > N_1$. Choose another positive integer N_2 such that $|\xi_k| < \frac{\epsilon}{2}$ for all $k > N_2$. By the definition of T , we have $|\langle x_n^*, x_n \rangle| < \epsilon$ for all $n > \max(N_1, N_2)$. Thus $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$.

(3) \Rightarrow (4) is trivial.

(4) \Rightarrow (5). If $(x_n)_n$ is weakly p -summable in X and $(x_n^*)_n$ is weakly Cauchy in X^* , yet $(\langle x_n^*, x_n \rangle)_n$ does not converge to 0. By passing to subsequences, we may

assume that $|\langle x_n^*, x_n \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and all $n \in \mathbb{N}$. Since $(x_n)_n$ is weakly p -summable and in particular weakly null, there exists a subsequence $(x_{k_n})_n$ of $(x_n)_n$ such that $|\langle x_n^*, x_{k_n} \rangle| < \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Since $(x_n^*)_n$ is weakly Cauchy, we see that $(x_{k_n}^* - x_n^*)_n$ is weakly null. By (3), $\lim_{n \rightarrow \infty} \langle x_{k_n}^* - x_n^*, x_{k_n} \rangle = 0$. This implies that $|\langle x_{k_n}^* - x_n^*, x_{k_n} \rangle| < \frac{\epsilon_0}{3}$ for n large enough. But for such n 's, we have

$$\epsilon_0 < |\langle x_{k_n}^*, x_{k_n} \rangle| \leq |\langle x_{k_n}^* - x_n^*, x_{k_n} \rangle| + |\langle x_n^*, x_{k_n} \rangle| < \frac{5\epsilon_0}{6}.$$

(5) \Rightarrow (1). Let $T : X \rightarrow Y$ be a weakly compact operator. Let us suppose that T is not unconditionally p -converging. Appealing again to Theorem 2.2, we obtain a weakly p -summable sequence $(x_n)_n$ in X and $\epsilon_0 > 0$ such that $\|Tx_n\| > \epsilon_0$ ($n = 1, 2, \dots$). Pick $y_n^* \in Y^*$ such that $\langle y_n^*, Tx_n \rangle = \|Tx_n\|$ and $\|y_n^*\| = 1$ for all $n \in \mathbb{N}$. Since T is weakly compact, so is T^* . Hence there is a subsequence $(y_{k_n}^*)_n$ of $(y_n^*)_n$ such that the sequence $(T^*y_{k_n}^*)_n$ converges weakly and hence is weakly Cauchy. The assumption ensures that the sequence $(\langle T^*y_{k_n}^*, x_{k_n} \rangle)_n = (\|Tx_{k_n}\|)_n$ converges to 0, which is a contradiction. □

Corollary 3.3. *Let $1 < p < \infty$. If X^{**} has the DPP_p , then so is X .*

The converse of Corollary 3.3 is not true. In fact, the Banach space $X = (\sum_n l_2^n)_{c_0}$ enjoys the DPP, but $X^{**} = (\sum_n l_2^n)_{l_\infty}$ contains a complemented copy of l_2 . Since l_2 fails the DPP_p for any $2 \leq p < \infty$, X^{**} also fails the DPP_p for any $2 \leq p < \infty$. In the case of the classical DPP, there is a result better than Corollary 3.3: If X^* has the DPP, then X has the DPP too (see [10]). The analogous result is not true for the DPP_p : for each $1 < p < \infty$, every operator from l_p into Tsirelson's space T is compact, hence T has the DPP_p for any $1 < p < \infty$. But, for each $1 < p < \infty$, there is a non-compact operator from l_p into T^* . Thus, for each $1 < p < \infty$, T^* fails the DPP_p .

Corollary 3.4. *Suppose that a Banach space X contains no copy of l_1 and let $1 < p < \infty$. The following statements are equivalent:*

- (1) X^* has the DPP_p ;
- (2) For all Banach spaces Y , every weakly compact operator $T : Y \rightarrow X$ has the unconditionally p -converging adjoint;
- (3) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n^*)_n \in l_p^w(X^*)$ and every weakly Cauchy sequence $(x_n)_n$ in X ;

- (4) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every weakly p -Cauchy sequence $(x_n^*)_n$ in X^* and every weakly null sequence $(x_n)_n$ in X ;
- (5) $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n^*)_n \in l_p^w(X^*)$ and every weakly null sequence $(x_n)_n$ in X .

Proof. We only prove (2) \Rightarrow (3) and (5) \Rightarrow (1).

(2) \Rightarrow (3). Assuming the contrary, we can find $(x_n^*)_n \in l_p^w(X^*)$ and a weakly Cauchy sequence $(x_n)_n$ in X such that $|\langle x_n^*, x_n \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and all $n \in \mathbb{N}$. Since $(x_n^*)_n$ is weakly null, there exists a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ such that $|\langle x_{k_n}^*, x_n \rangle| < \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Thus $|\langle x_{k_n}^*, x_n - x_{k_n} \rangle| > \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Define an operator $S : X^* \rightarrow c_0$ by

$$Sx^* = (\langle x^*, x_n - x_{k_n} \rangle)_n, \quad x^* \in X^*.$$

It is easy to check that $S^*e_n = x_n - x_{k_n}$ ($n = 1, 2, \dots$), where $(e_n)_n$ is the unit vector basis of l_1 . Thus the operator S^* maps l_1 into X and is weakly compact. By (2), the operator S^{**} is unconditionally p -converging. Moreover, an easy verification shows that $S^{**} = S$. By Theorem 2.2, we get $\lim_{n \rightarrow \infty} \|Sx_{k_n}^*\| = 0$. It follows from the definition of the operator S that $\lim_{n \rightarrow \infty} |\langle x_{k_n}^*, x_n - x_{k_n} \rangle| = 0$, which is a contradiction.

(5) \Rightarrow (1). By Theorem 3.2, it is enough to verify that for every $(x_n^*)_n \in l_p^w(X^*)$ and every weakly null sequence $(x_n^{**})_n$ in X^{**} , the sequence $(\langle x_n^{**}, x_n^* \rangle)_n$ converges to 0. Now we suppose that it is false. Then, by passing to subsequences, we may assume that $|\langle x_n^{**}, x_n^* \rangle| > \epsilon_0$ for some $\epsilon_0 > 0$ and all $n \in \mathbb{N}$. Of course, we may also assume that $\|x_n^{**}\| \leq 1$ for all $n \in \mathbb{N}$. It follows from Goldstine's Theorem that for each $n \in \mathbb{N}$, there exists an $x_n \in B_X$ such that $|\langle x_n - x_n^{**}, x_n^* \rangle| < \frac{\epsilon_0}{2}$. Then $|\langle x_n^*, x_n \rangle| > \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. By Rosenthal's Theorem, $(x_n)_n$ has a weakly Cauchy subsequence, which is still denoted by $(x_n)_n$. Then there exists a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ such that $|\langle x_{k_n}^*, x_n \rangle| < \frac{\epsilon_0}{3}$ for all $n \in \mathbb{N}$. By (5), we get $\lim_{n \rightarrow \infty} \langle x_{k_n}^*, x_{k_n} - x_n \rangle = 0$, which implies that $|\langle x_{k_n}^*, x_{k_n} - x_n \rangle| < \frac{\epsilon_0}{6}$ for n large enough. It is easy to verify that for such n 's, $|\langle x_{k_n}^*, x_{k_n} \rangle| < \frac{\epsilon_0}{2}$. This contradiction completes the proof. \square

Definition 3.1. Let $1 < p < \infty$. We say that a Banach space X has the *hereditary Dunford-Pettis property of order p* (in short, hereditary DPP_p) if every (closed) subspace of X has the DPP_p .

We present a useful characterization of hereditary DPP_p . We need a J. Elton's result that can be found in [11].

Lemma 3.5. [11] *If $(x_n)_n$ is a normalized weakly null sequence of a space X such that no subsequence of it is equivalent to the unit vector basis $(e_n)_n$ of c_0 , then $(x_n)_n$ has a subsequence $(y_n)_n$ for which given any subsequence $(z_n)_n$ of $(y_n)_n$ and any sequence $(\alpha_n)_n \in c_0$ we have $\sup_n \|\sum_{k=1}^n \alpha_k z_k\| = +\infty$.*

Theorem 3.6. *Let X be Banach space and $1 < p < \infty$. The following are equivalent:*

- (1) *X has the hereditary DPP_p ;*
- (2) *Every normalized weakly p -summable sequence in X admits a subsequence that is equivalent to the unit vector basis of c_0 ;*
- (3) *Every weakly p -summable sequence in X admits a weakly 1-summable subsequence;*
- (4) *Every weakly p -summable sequence in X admits a subsequence $(y_n)_n$ such that $\sup_N \|\sum_{n=1}^N y_n\| < \infty$.*

Proof. (1) \Rightarrow (2). Let $(x_n)_n$ be a normalized weakly p -summable sequence in X such that it admits no subsequence that is equivalent to the unit vector basis $(e_n)_n$ of c_0 . It follows from Lemma 3.5 that $(x_n)_n$ has a subsequence $(y_n)_n$ as stated in Lemma 3.5. By Bessaga-Pełczyński Selection Principle, we may assume that $(y_n)_n$ is a basic sequence. Let $X_0 = \overline{\text{span}}\{y_n : n = 1, 2, \dots\}$. Let $(y_n^*)_n \subset X_0^*$ be the coefficient functionals of the basic sequence $(y_n)_n$. For each N , define a projection $P_N : X_0 \rightarrow X_0$ by

$$P_N(y) = \sum_{n=1}^N \langle y_n^*, y \rangle y_n, \quad y \in X_0.$$

Then the projection P_N 's are uniformly bounded in operator norm. An easy verification shows that $P_N^{**}y^{**} = \sum_{n=1}^N \langle y^{**}, y_n^* \rangle y_n$ for all $y^{**} \in X_0^{**}$. Lemma 3.5 and the uniform boundedness of the projection P_N 's imply that $(\langle y^{**}, y_n^* \rangle)_n \in c_0$ for all $y^{**} \in X_0^{**}$, that is, $(y_n^*)_n$ is weakly null. Since $\langle y_n^*, y_n \rangle = 1$ for all $n \in \mathbb{N}$, it follows from Theorem 3.2 again that X_0 fails the DPP_p .

(2) \Rightarrow (3) and (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (1). Take a subspace X_0 of X that fails the DPP_p . Appealing to Theorem 3.2, we obtain a weakly compact operator $T : X_0 \rightarrow c_0$ which is not unconditionally p -converging. Applying Theorem 2.2, we get a normalized weakly p -summable sequence $(x_n)_n$ in X such that $\|Tx_n\| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Bessaga-Pełczyński Selection

Principle allows us to assume that the sequence $(Tx_n)_n$ is equivalent to the unit vector basis $(e_n)_n$ of c_0 . By the weak compactness of T , the sequence $(x_n)_n$ admits no subsequence equivalent to the unit vector basis $(e_n)_n$. By Lemma 3.5, the sequence $(x_n)_n$ admits a subsequence $(y_n)_n$ for which given any subsequence $(z_n)_n$ of $(y_n)_n$, one has $\sup_N \|\sum_{n=1}^N z_n\| = \infty$.

□

A direct consequence of Theorem 3.6 is the following corollary:

Corollary 3.7. *If a Banach space X has the hereditary DPP_p , then each weakly p -summable sequence in X admits a subsequence $(x_n)_n$ such that $\lim_{n \rightarrow \infty} \|\sum_{k=1}^n x_k\|/n^{\frac{1}{p^*}} = 0$.*

We close this section with the surjective DPP_p , a formally weaker property than the DPP_p . By the Davis-Figiel-Johnson-Pełczyński's factorization theorem (see [9]), a Banach space X has the DPP_p if and only if for all reflexive spaces Y , every operator from X into Y is unconditionally p -converging. We introduce the surjective DPP_p by imposing that every surjective operator from X onto the reflexive space Y is unconditionally p -converging. The motivation for introducing the surjective DPP_p was to extend the surjective DPP introduced in [21].

Definition 3.2. Let $1 < p < \infty$. We say that a Banach space X has the *surjective DPP_p* if for all reflexive spaces Y , every surjective operator from X onto Y is unconditionally p -converging.

The following are the internal characterizations of the surjective DPP_p .

Theorem 3.8. *The following are equivalent for a Banach space X and $1 < p < \infty$:*

- (1) *X has the surjective DPP_p ;*
- (2) *$\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every weakly p -Cauchy sequence $(x_n)_n$ in X and every weakly null sequence $(x_n^*)_n$ in X^* such that $\overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$ is reflexive;*
- (3) *$\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n)_n \in l_p^w(X)$ and every weakly null sequence $(x_n^*)_n$ in X^* such that $\overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$ is reflexive;*
- (4) *$\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, for every $(x_n)_n \in l_p^w(X)$ and every weakly Cauchy sequence $(x_n^*)_n$ in X^* such that $\overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$ is reflexive.*

Proof. (1) \Rightarrow (2). Let $(x_n)_n \subset X$ and $(x_n^*)_n \subset X^*$ be as in (2). Let $Z = \overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$. Then $(Z_\perp)^\perp = Z$, where $Z_\perp := \{x \in X : \langle x^*, x \rangle = 0 \text{ for all } x^* \in Z\}$

and $(Z_\perp)^\perp := \{x^* \in X^* : \langle x^*, x \rangle = 0 \text{ for all } x \in Z_\perp\}$. Let $Q : X \rightarrow X/Z_\perp$ be the natural quotient. Then $Q^* : (X/Z_\perp)^* \rightarrow Z$ is a surjective isometrical isomorphism. Let $Q^* f_n = x_n^*$, $f_n \in (X/Z_\perp)^*$ for all $n \in \mathbb{N}$. By (1), the quotient Q is unconditionally p -converging. By Theorem 2.2, the sequence $(Qx_n)_n$ converges in norm to Qx for some $x \in X$. Thus

$$|\langle x_n^*, x_n - x \rangle| = |\langle f_n, Qx_n - Qx \rangle| \leq (\sup_n \|f_n\|) \|Qx_n - Qx\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Since $(x_n^*)_n$ is weakly null, $\lim_{n \rightarrow \infty} \langle x_n^*, x \rangle = 0$. Therefore we have $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$.

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (4). Suppose that (4) is false. Then there exist a sequences $(x_n)_n \in l_p^w(X)$ and a weakly Cauchy sequence $(x_n^*)_n$ in X^* such that $\overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$ is reflexive so that $|\langle x_n^*, x_n \rangle| > \epsilon_0 > 0$ for all $n \in \mathbb{N}$. Since the sequence $(x_n)_n$ converges to 0 weakly, there is a subsequence $(x_{k_n})_n$ of $(x_n)_n$ such that $|\langle x_n^*, x_{k_n} \rangle| < \frac{\epsilon_0}{2}$ for all $n \in \mathbb{N}$. Since the space $\overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$ is reflexive, the space $\overline{\text{span}}\{x_n^* - x_{k_n}^* : n = 1, 2, \dots\}$ is reflexive too. By the hypothesis, $\lim_{n \rightarrow \infty} \langle x_n^* - x_{k_n}^*, x_{k_n} \rangle = 0$. Thus, $|\langle x_n^* - x_{k_n}^*, x_{k_n} \rangle| < \frac{\epsilon_0}{2}$ for n large enough, which implies that for such n 's, $|\langle x_{k_n}^*, x_{k_n} \rangle| < \epsilon_0$, a contradiction.

(4) \Rightarrow (1). Suppose that X fails the surjective DPP_p . Then there exists a surjective operator T from X onto a reflexive space Y such that T is not unconditionally p -converging. By Theorem 2.2, there exists a normalized weakly p -summable sequence $(x_n)_n$ in X such that $\|Tx_n\| > \epsilon_0$ for all $n \in \mathbb{N}$. For each n , choose $y_n^* \in Y^*$ with $\|y_n^*\| = 1$ such that $\langle y_n^*, Tx_n \rangle = \|Tx_n\|$. By the reflexivity of Y , we may assume that the sequence $(y_n^*)_n$ converges to 0 weakly by passing to subsequences if necessary. Let $x_n^* = T^* y_n^*$. Then the sequence $(x_n^*)_n$ converges to 0 weakly too. Since T is surjective, the operator $T^* : Y^* \rightarrow X^*$ is an isomorphic embedding. This implies that the space $\overline{\text{span}}\{x_n^* : n = 1, 2, \dots\}$ is contained in $T^*(\overline{\text{span}}\{y_n^* : n = 1, 2, \dots\})$ and hence is reflexive. By (4), $\lim_{n \rightarrow \infty} \langle x_n^*, x_n \rangle = 0$, a contradiction because $\langle x_n^*, x_n \rangle > \epsilon_0$ for all $n \in \mathbb{N}$. This concludes the proof. □

An immediate consequence of Theorem 3.8 is the following:

Corollary 3.9. *Let $1 < p < \infty$. If X^{**} has the surjective DPP_p , then so is X .*

We also use the space $X = (\sum_n l_2^n)_{c_0}$ to show that the converse of Corollary 3.9 is not true. The same argument shows that the space $X = (\sum_n l_2^n)_{c_0}$ enjoys the surjective DPP_p for any $1 < p < \infty$, but X^{**} also fails the surjective DPP_p for any $2 \leq p < \infty$.

The following result analogous to Theorem 3 in [4] shows that the surjective DPP_p and the DPP_p coincide for certain classes of Banach spaces.

Theorem 3.10. *If a Banach space X contains a complemented copy of l_1 , then X has the DPP_p if and only if X has the surjective DPP_p .*

4. QUANTIFYING UNCONDITIONALLY p -CONVERGING OPERATORS

As discussed above, we see that unconditionally p -converging operators are intermediate between completely continuous operators and unconditionally converging operators. Precisely, we have the following implications:

$$T \text{ completely continuous} \Rightarrow T \text{ unconditionally } p\text{-converging} \Rightarrow T \text{ unconditionally converging.}$$

In this section, we quantify these implications. We need some necessary quantities.

Let $(x_n)_n$ be a bounded sequence in a Banach space X . Set

$$ca((x_n)_n) = \inf_n \sup \{\|x_k - x_l\| : k, l \geq n\}.$$

This quantity is a measure of non-Cauchyness of the sequence $(x_n)_n$. More precisely, $ca((x_n)_n) = 0$ if and only if $(x_n)_n$ is norm Cauchy. In [17], an important quantity measuring how far an operator $T : X \rightarrow Y$ is from being completely continuous, denoted as $cc(T)$, is defined by

$$cc(T) = \sup \{ca((Tx_n)_n) : (x_n)_n \subset B_X \text{ weakly Cauchy}\}.$$

Obviously, T is completely continuous if and only if $cc(T) = 0$. In this note, we define another equivalent quantity measuring the complete continuity of an operator $T : X \rightarrow Y$ as follows:

$$cc_n(T) = \sup \{\limsup_n \|Tx_n\| : (x_n)_n \subset B_X \text{ weakly null}\}.$$

Obviously, T is completely continuous if and only if $cc_n(T) = 0$. The following theorem demonstrates these two quantities are equivalent.

Theorem 4.1. *Let $T \in \mathcal{L}(X, Y)$. Then $cc_n(T) \leq cc(T) \leq 2cc_n(T)$.*

To prove Theorem 4.1, we need the following lemma.

Lemma 4.2. *Let X be a Banach space and $(x_n)_n$ be a weakly null sequence in B_X . Let $\epsilon > 0$ be such that $\|x_n\| > \epsilon$ for all $n \in \mathbb{N}$. Then, for every $\delta > 0$, there is a subsequence $(x_{k_n})_n$ of $(x_n)_n$ such that $ca((x_{k_n})_n) \geq \epsilon - \delta$.*

Proof. We set $x_{k_1} = x_1$. Choose $x_1^* \in S_{X^*}$ such that $\langle x_1^*, x_{k_1} \rangle = \|x_{k_1}\|$. Since $(x_n)_n$ is weakly null, there exists $k_2 > k_1$ such that $|\langle x_1^*, x_{k_2} \rangle| < \delta$. Then

$$\|x_{k_1} - x_{k_2}\| \geq |\langle x_1^*, x_{k_1} - x_{k_2} \rangle| \geq |\langle x_1^*, x_{k_1} \rangle| - |\langle x_1^*, x_{k_2} \rangle| \geq \epsilon - \delta.$$

Suppose that we have obtained $\{x_{k_1}, x_{k_2}, \dots, x_{k_n}\}$ such that $\|x_{k_i} - x_{k_n}\| \geq \epsilon - \delta$ for $i = 1, 2, \dots, n-1$. Let $Y_n = \text{span}\{x_{k_1}, x_{k_2}, \dots, x_{k_n}\}$. Pick a c -net $\{z_1, z_2, \dots, z_m\} \subset S_{Y_n}$ for S_{Y_n} , where $0 < c < \frac{\delta}{2}$. Choose $z_1^*, z_2^*, \dots, z_m^*$ in S_{X^*} such that $\langle z_i^*, z_i \rangle = 1$ for $i = 1, 2, \dots, m$. Since $(x_n)_n$ is weakly null, there exists $k_{n+1} > k_n$ such that $|\langle z_i^*, x_{k_{n+1}} \rangle| < c$ for all $i = 1, 2, \dots, m$. Then, for each $1 \leq j \leq n$, there exists $1 \leq i \leq m$ such that $\|\frac{x_{k_j}}{\|x_{k_j}\|} - z_i\| < c$. Thus

$$\begin{aligned} \|x_{k_j} - x_{k_{n+1}}\| &\geq |\langle z_i^*, x_{k_j} - x_{k_{n+1}} \rangle| \\ &\geq 1 - |\langle z_i^*, x_{k_{n+1}} \rangle| - |\langle z_i^*, x_{k_j} - z_i \rangle| \\ &\geq 1 - c - \|x_{k_j} - z_i\| \\ &\geq 1 - c - (1 + c - \epsilon) = \epsilon - 2c \\ &\geq \epsilon - \delta \end{aligned}$$

By induction, we get a subsequence $(x_{k_n})_n$ such that $\|x_{k_n} - x_{k_m}\| \geq \epsilon - \delta$ ($n \neq m, n, m = 1, 2, \dots$). This yields that $ca((x_{k_n})_n) \geq \epsilon - \delta$. □

Proof of Theorem 4.1. Step 1. $cc(T) \leq 2cc_n(T)$.

We may suppose that $cc(T) > 0$ and fix any $c > 0$ satisfying $cc(T) > c$. Then there is a weakly Cauchy sequence $(x_n)_n$ in B_X such that $ca((Tx_n)_n) > c$. It follows that there exist two strictly increasing sequences $(k_n)_n, (l_n)_n$ of positive integers such that $\|Tx_{k_n} - Tx_{l_n}\| > c$ for all $n \in \mathbb{N}$. Set $z_n = (x_{k_n} - x_{l_n})/2$. Then $(z_n)_n$ is a weakly null sequence in B_X and $\|Tz_n\| > c/2$ for each $n \in \mathbb{N}$. Hence $\limsup_n \|Tz_n\| \geq c/2$ and then $cc_n(T) \geq c/2$. Since $c < cc(T)$ is arbitrary, we get $cc(T) \leq 2cc_n(T)$.

Step 2. $cc_n(T) \leq cc(T)$.

We may suppose that $\|T\| = 1$ and $cc_n(T) > 0$. Suppose that $cc_n(T) > \epsilon > 0$. Then there is a weakly null sequence $(x_n)_n$ in B_X such that $\limsup_n \|Tx_n\| > \epsilon$. This

yields a subsequence of $(x_n)_n$, still denoted by $(x_n)_n$, so that $\|Tx_n\| > \epsilon$ for each $n \in \mathbb{N}$. By Lemma 4.2, for every $\delta > 0$, there is a subsequence $(x_{k_n})_n$ of $(x_n)_n$ such that $ca((Tx_{k_n})_n) \geq \epsilon - \delta$. This means that $cc(T) \geq \epsilon - \delta$. Since $\delta > 0$ is arbitrary, we get $cc(T) \geq \epsilon$. By the arbitrariness of $\epsilon < cc_n(T)$, we obtain $cc_n(T) \leq cc(T)$. This completes the proof of Theorem 4.1. \square

To quantify unconditionally p -converging operators, we will need two measures of non-compactness. Let us fix some notations. If A and B are nonempty subsets of a Banach space X , we set

$$d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\},$$

$$\widehat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

Thus, $d(A, B)$ is the ordinary distance between A and B , and $\widehat{d}(A, B)$ is the non-symmetrized Hausdorff distance from A to B .

Let A be a bounded subset of a Banach space X . The Hausdorff measure of non-compactness of A is defined by

$$\chi(A) = \inf\{\widehat{d}(A, F) : F \subset X \text{ finite}\},$$

$$\chi_0(A) = \inf\{\widehat{d}(A, F) : F \subset A \text{ finite}\}.$$

Then $\chi(A) = \chi_0(A) = 0$ if and only if A is relatively norm compact. It is easy to verify that

$$(4.1) \quad \chi(A) \leq \chi_0(A) \leq 2\chi(A).$$

Now we define five quantities which measure how far an operator is from being unconditionally p -converging. Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p < \infty$. We set

$$uc_p^1(T) = \sup\{\limsup_n \|Tx_n\| : (x_n)_n \in l_p^w(X), (x_n)_n \subset B_X\},$$

$$uc_p^2(T) = \sup\{ca((Tx_n)_n) : (x_n)_n \subset B_X \text{ weakly } p\text{-Cauchy}\},$$

$$uc_p^3(T) = \sup\{ca((Tx_n)_n) : (x_n)_n \subset B_X \text{ weakly } p\text{-convergent}\},$$

$$uc_p^4(T) = \sup\{\chi_0(TL) : L \subset B_X \text{ relatively weakly } p\text{-compact}\},$$

$$uc_p^5(T) = \sup\{\chi_0(TL) : L \subset B_X \text{ relatively weakly } p\text{-precompact}\}.$$

Clearly, $uc_p^1(T) = uc_p^2(T) = uc_p^3(T) = uc_p^4(T) = uc_p^5(T) = 0$ if and only if T is unconditionally p -converging. It turns out that the above five quantities are equivalent.

Theorem 4.3. *Let $T \in \mathcal{L}(X, Y)$ and $1 < p < \infty$. Then*

$$uc_p^5(T) \leq uc_p^3(T) \leq uc_p^2(T) \leq 2uc_p^1(T) \leq 2uc_p^4(T) \leq 2uc_p^5(T).$$

Proof. Step 1. $uc_p^5(T) \leq uc_p^3(T)$.

We may assume that $uc_p^5(T) > 0$. Let us fix any $0 < c < uc_p^5(T)$. Then there exists a relatively weakly p -precompact subset $L \subset B_X$ such that $\chi_0(TL) > c$. By induction, we can construct a sequence $(x_n)_n$ in L such that $\|Tx_n - Tx_m\| > c, n \neq m, n, m = 1, 2, \dots$. Since L is relatively weakly p -precompact, the sequence $(x_n)_n$ admits a weakly p -convergent subsequence that is still denoted by $(x_n)_n$. Thus we get $ca((Tx_n)_n) \geq c$, which yields $uc_p^3(T) \geq c$. By the arbitrariness of c , we get $uc_p^5(T) \leq uc_p^3(T)$.

Step 2. $uc_p^2(T) \leq 2uc_p^1(T)$.

We assume that $uc_p^2(T) > 0$ and fix any $0 < c < uc_p^2(T)$. Then there is a weakly p -Cauchy sequence $(x_n)_n$ in B_X such that $ca((Tx_n)_n) > c$. By induction, there exist two strictly increasing sequences $(k_n)_n, (l_n)_n$ of positive integers such that $\|Tx_{k_n} - Tx_{l_n}\| > c$ for all $n \in \mathbb{N}$. Set $z_n = (x_{k_n} - x_{l_n})/2$. Then $(z_n)_n$ is a weakly p -summable sequence in B_X and $\|Tz_n\| > c/2$ for each $n \in \mathbb{N}$. Hence $uc_p^1(T) \geq c/2$. Since c is arbitrary, we get Step 2.

Step 3. $uc_p^1(T) \leq uc_p^4(T)$.

Suppose $uc_p^1(T) > c > 0$. Then there exists a weakly p -summable sequence $(x_n)_n$ in B_X such that $\|Tx_n\| > c$ for all $n \in \mathbb{N}$. We claim that $\chi_0((Tx_n)_n) \geq c$. If this is false, we can find a finite subset F of $(Tx_n)_n$ such that $\widehat{d}((Tx_n)_n, F) < c$. Since F is finite, there exist $y \in F$ and a subsequence $(Tx_{k_n})_n$ of $(Tx_n)_n$ such that $\|Tx_{k_n} - y\| \leq c$ for each $n \in \mathbb{N}$. Since the sequence $(Tx_{k_n})_n$ is weakly null, we get $\|y\| \leq c$. This contradiction completes the proof Step 3.

The remaining inequalities $uc_p^3(T) \leq uc_p^2(T), uc_p^4(T) \leq uc_p^5(T)$ are immediate. □

It should be mentioned that a quantity is defined in [20] to measure how far an operator is unconditionally converging as follows:

$$uc(T) = \sup\{ca\left(\sum_{i=1}^n Tx_i\right) : (x_n)_n \in l_1^w(X), \|(x_n)_n\|_1^w \leq 1\}.$$

Obviously, $uc(T) = 0$ if and only if T is unconditionally converging. Inspired by this quantity, we define the sixth quantity measuring how far an operator is unconditionally p -converging as follows:

$$uc_p^6(T) = \sup\{\limsup_n \|Tx_n\| : (x_n)_n \in l_p^w(X), \|(x_n)_n\|_p^w \leq 1\}.$$

It is obvious that $uc_p^6(T) = 0$ if and only if T is unconditionally p -converging. This new quantity will be used in next section to prove a quantitative version of the Dunford-Pettis property of order p .

Theorem 4.4. *Let $T \in \mathcal{L}(X, Y)$. Then $uc_1^6(T) = uc(T)$.*

Proof. Step 1. $uc_1^6(T) \leq uc(T)$.

Let $(x_n)_n \in l_1^w(X)$ with $\|(x_n)_n\|_1^w \leq 1$. It aims to show $\limsup_n \|Tx_n\| \leq ca((\sum_{i=1}^n Tx_i)_n)$. Let $c > ca((\sum_{i=1}^n Tx_i)_n)$. Then there exists $n \in \mathbb{N}$ such that $\|\sum_{i=1}^k Tx_i - \sum_{i=1}^l Tx_i\| < c$ for all $k, l \geq n$. In particular, we have $\|Tx_k\| = \|\sum_{i=1}^k Tx_i - \sum_{i=1}^{k-1} Tx_i\| < c$ for all $k \geq n+1$. Thus one can derive that $\limsup_n \|Tx_n\| \leq c$. Since $c > ca((\sum_{i=1}^n Tx_i)_n)$ is arbitrary, we get $\limsup_n \|Tx_n\| \leq ca((\sum_{i=1}^n Tx_i)_n)$.

Step 2. $uc(T) \leq uc_1^6(T)$.

We can suppose that $uc(T) > 0$ and fix an arbitrary $0 < c < uc(T)$. Then there exists $(x_n)_n \in l_1^w(X)$ with $\|(x_n)_n\|_1^w \leq 1$ such that $ca((\sum_{i=1}^n Tx_i)_n) > c$. By induction, we can find two strictly increasing sequences $(k_n)_n, (l_n)_n, l_n < k_n$ of positive integers such that $\|\sum_{i=l_n+1}^{k_n} Tx_i\| > c$ for all $n \in \mathbb{N}$. Let $z_n = \sum_{i=l_n+1}^{k_n} x_i (n = 1, 2, \dots)$. It is easy to see that $(z_n)_n$ belongs to $l_1^w(X)$ with $\|(z_n)_n\|_1^w \leq 1$ such that $\|Tz_n\| > c$ for all $n \in \mathbb{N}$, which yields $\limsup_n \|Tz_n\| \geq c$. Hence $uc_1^6(T) \geq c$ and the proof of Step 2 is completed. □

Combining Theorem 4.1 with Theorem 4.4, we get the promised quantitative versions of the above implications.

Theorem 4.5. *Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p < \infty$. Then $uc(T) \leq uc_p^6(T) \leq cc(T)$.*

5. QUANTIFYING DUNFORD-PETTIS PROPERTY OF ORDER p

Let X be a Banach space and let \mathcal{F} be the family of all weakly compact subsets of B_{X^*} . For $F \in \mathcal{F}$, define a semi-norm q_F on X^{**} by

$$q_F(x^{**}) = \sup_{x^* \in F} |\langle x^{**}, x^* \rangle|, \quad x^{**} \in X^{**}.$$

The locally convex topology generated by the family of semi-norms $\{q_F : F \in \mathcal{F}\}$ is called the Mackey topology, denoted by $\tau(X^{**}, X^*)$. The restriction to X of the Mackey topology $\tau(X^{**}, X^*)$ is called the Right topology in [23]. This topology is denoted by ρ_X or simply ρ when X is obvious.

In this section, we introduce a new locally convex topology. Let X be a Banach space and let $1 \leq p < \infty$. Let \mathcal{F}_p be the family of all relatively weakly p -compact subsets of X . For $F \in \mathcal{F}_p$, we define a semi-norm q_F on X^* by

$$q_F(x^*) = \sup_{x \in F} |\langle x^*, x \rangle|, \quad x^* \in X^*.$$

The locally convex topology generated by the family of semi-norms $\{q_F : F \in \mathcal{F}_p\}$ is denoted by ρ_p^* when X is obvious. Applying Grothendieck's Completeness Theorem ([24, p.148]), we obtain that the space (X^*, ρ_p^*) is complete. Hence, a bounded subset A of X^* is relatively ρ_p^* -compact if and only if A is totally bounded, equivalently, the set $A|_F = \{x^*|_F : x^* \in A\}$ is totally bounded in $l_\infty(F)$ for each relatively weakly p -compact subset $F \subset B_X$. So, if we set

$$\chi_m^p(A) = \sup\{\chi_0(A|_F) : F \in \mathcal{F}_p, F \subset B_X\},$$

then A is relatively ρ_p^* -compact if and only if $\chi_m^p(A) = 0$. The following result, which is immediate from [19, Lemma 4.4], implies that an operator $T : X \rightarrow Y$ is unconditionally p -converging if and only if $T^*B_{Y^*}$ is relatively ρ_p^* -compact.

Theorem 5.1. *Let $T \in \mathcal{L}(X, Y)$ and $1 \leq p < \infty$. Then $\frac{1}{2}uc_p^4(T) \leq \chi_m^p(T^*B_{Y^*}) \leq 2uc_p^4(T)$.*

Let $(x_n^*)_n$ be a bounded sequence in X^* . We set

$$ca_{\mathcal{F}_p}((x_n^*)_n) = \sup_{F \in \mathcal{F}_p, F \subset B_X} \inf_n \sup \{q_F(x_k^* - x_l^*) : k, l \geq n\},$$

and

$$\tilde{ca}_{\mathcal{F}_p}((x_n^*)_n) = \inf\{ca_{\mathcal{F}_p}((x_{k_n}^*)_n) : (x_{k_n}^*)_n \text{ is a subsequence of } (x_n^*)_n\}.$$

The quantity $ca_{\mathcal{F}_p}$ measures how far the sequence $(x_n^*)_n$ is from being ρ_p^* -Cauchy. In particular, $ca_{\mathcal{F}_p}((x_n^*)_n) = 0$ if and only if the sequence $(x_n^*)_n$ is ρ_p^* -Cauchy.

The following result contains two topological characterizations of DPP_p .

Theorem 5.2. *The following are equivalent about a Banach space X and $1 < p < \infty$:*

- (1) X has the DPP_p ;
- (2) Every weakly p -summable sequence in X is ρ -null;
- (3) Every weakly convergent sequence in X^* is ρ_p^* -convergent.

Proof. The equivalence of (1) and (2) is essentially Theorem 3.1. The implication (3) \Rightarrow (1) follows from Theorem 3.2. It remains to prove (1) \Rightarrow (3).

Let $(x_n^*)_n$ be weakly null in X^* . Define an operator $T : X \rightarrow c_0$ by

$$Tx = (\langle x_n^*, x \rangle)_n, \quad x \in X.$$

Since $(x_n^*)_n$ is weakly null, T is weakly compact. By (1), we get T is unconditionally p -converging. Let $F \in \mathcal{F}_p$. It follows from Theorem 2.3 that TF is relatively norm compact in c_0 . By the well-known characterization of relatively norm compact subsets of c_0 , we get

$$\lim_{n \rightarrow \infty} q_F(x_n^*) = \lim_{n \rightarrow \infty} \sup_{x \in F} |\langle x_n^*, x \rangle| = 0,$$

which implies that $(x_n^*)_n$ is ρ_p^* -null. □

To quantify the DPP_p , we will need several measures of weak non-compactness. Let A be a bounded subset of a Banach space X . The de Blasi measure of weak non-compactness of A is defined by

$$\omega(A) = \inf \{ \widehat{d}(A, K) : \emptyset \neq K \subset X \text{ is weakly compact} \}.$$

Then $\omega(A) = 0$ if and only if A is relatively weakly compact. It is easy to verify that

$$\omega(A) = \inf \{ \epsilon > 0 : \text{there exists a weakly compact subset } K \text{ of } X \text{ such that } A \subset K + \epsilon B_X \}.$$

Other commonly used quantities measuring weak non-compactness are:

$$wk_X(A) = \widehat{d}(\overline{A}^{w*}, X), \text{ where } \overline{A}^{w*} \text{ denotes the weak}^* \text{ closure of } A \text{ in } X^{**}.$$

$$wck_X(A) = \sup \{ d(clust_{X^{**}}((x_n)_n), X) : (x_n)_n \text{ is a sequence in } A \}, \text{ where}$$

$$clust_{X^{**}}((x_n)_n) \text{ is the set of all weak}^* \text{ cluster points in } X^{**} \text{ of } (x_n)_n.$$

$$\gamma(A) = \sup \{ |\lim_n \lim_m \langle x_m^*, x_n \rangle - \lim_m \lim_n \langle x_m^*, x_n \rangle| : (x_n)_n \text{ is a sequence in } A, (x_m^*)_m \text{ is a sequence in } B_{X^*} \text{ and all the involved limits exist} \}.$$

It follows from [1, Theorem 2.3] that for any bounded subset A of a Banach space X we have

$$wck_X(A) \leq wk_X(A) \leq \gamma(A) \leq 2wck_X(A),$$

$$wk_X(A) \leq \omega(A).$$

For an operator T , $\omega(T)$, $wk_Y(T)$, $wck_Y(T)$, $\gamma(T)$ denote $\omega(TB_X)$, $wk_Y(TB_X)$, $wck_Y(TB_X)$ and $\gamma(TB_X)$, respectively. C. Angosto and B. Cascales([1]) proved the following inequality:

$$\gamma(T) \leq \gamma(T^*) \leq 2\gamma(T), \text{ for any operator } T.$$

So, putting these inequalities together, we get, for any operator T ,

$$(5.1) \quad \frac{1}{2}wk_Y(T) \leq wk_{X^*}(T^*) \leq 4wk_Y(T).$$

Let X be a Banach space and A be a bounded subset of X^* . For $1 \leq p < \infty$, we set

$$\iota_p(A) = \sup \left\{ \limsup_n \sup_{x^* \in A} | \langle x^*, x_n \rangle | : (x_n)_n \in l_p^w(X), (x_n)_n \subset B_X \right\},$$

$$\eta_p(A) = \sup \left\{ \limsup_n \sup_{x^* \in A} | \langle x^*, x_n \rangle | : (x_n)_n \in l_p^w(X), \|(x_n)_n\|_p^w \leq 1 \right\}.$$

These two quantities measure how far A is weakly p -limited. Obviously, $\eta_p(A) = \iota_p(A) = 0$ if and only if A is weakly p -limited. The following theorem says, in particular, that weakly p -limited sets coincide with relatively ρ_p^* -compact sets. Its proof is similar to [19, Lemma 5.6].

Theorem 5.3. *Let X be a Banach space, $1 \leq p < \infty$ and A be a bounded subset of X^* . Then*

$$\frac{1}{8}\chi_m^p(A) \leq \iota_p(A) \leq \chi_m^p(A).$$

In the following theorem, we quantify the DPP_p by using the quantities $\omega(\cdot)$, $\iota_p(\cdot)$, $\tilde{c}a_{\mathcal{F}_p}(\cdot)$ and $\chi_m^p(\cdot)$.

Theorem 5.4. *Let X be a Banach space and $1 < p < \infty$. The following are equivalent:*

- (1) X has the DPP_p ;
- (2) $uc_p^1(T) \leq \omega(T^*)$ for every operator T from X into any Banach space Y ;
- (3) $\iota_p(A) \leq \omega(A)$ for every bounded subset A of X^* ;
- (4) $\tilde{c}a_{\mathcal{F}_p}((x_n^*)_n) \leq 2\omega((x_n^*)_n)$ whenever $(x_n^*)_n$ is a bounded sequence in X^* ;
- (5) $\chi_m^p(A) \leq 2\omega(A)$ for every bounded subset A of X^* .

Proof. (2) \Rightarrow (1) is obvious. (3) \Rightarrow (1) and (5) \Rightarrow (1) follow from Theorem 3.1.

(1) \Rightarrow (2). Let Y be a Banach space and let $T \in \mathcal{L}(X, Y)$. Let $\epsilon > 0$ be such that $T^*B_{Y^*} \subset K + \epsilon B_{X^*}$, $K \subset X^*$ is weakly compact. Let $(x_n)_n \in l_p^w(X)$ and $(x_n)_n \subset B_X$. Since X has the DPP_p , it follows from Theorem 3.1 that $\lim_{n \rightarrow \infty} \sup_{x^* \in K} | \langle x^*, x_n \rangle | = 0$. Let $c > 0$. Then there exists a positive integer N such that $\sup_{x^* \in K} | \langle x^*, x_n \rangle | < c$ for each $n \geq N$. For each $n \in \mathbb{N}$, pick $y_n^* \in B_{Y^*}$ with $\|Tx_n\| = \langle y_n^*, Tx_n \rangle$. Since $T^*B_{Y^*} \subset K + \epsilon B_{X^*}$, then, for each $n \in \mathbb{N}$, there exists $x_n^* \in K$ such that

$\|T^*y_n^* - x_n^*\| \leq \epsilon$. Then, for $n \geq N$, we get

$$\begin{aligned} \|Tx_n\| &= \langle T^*y_n^*, x_n \rangle \\ &\leq \epsilon + |\langle x_n^*, x_n \rangle| \\ &\leq \epsilon + \sup_{x^* \in K} |\langle x^*, x_n \rangle| \\ &\leq \epsilon + c. \end{aligned}$$

This yields $\limsup_n \|Tx_n\| \leq \epsilon + c$. Since $c > 0$ is arbitrary, we obtain $\limsup_n \|Tx_n\| \leq \epsilon$ and hence $uc_p^1(T) \leq \epsilon$. This proves $uc_p^1(T) \leq \omega(T^*)$.

(1) \Rightarrow (3). Let $(x_n)_n$ be a weakly p -summable sequence in B_X . Let $\epsilon > 0$ be such that $A \subset K + \epsilon B_{X^*}$, $K \subset X^*$ is weakly compact. For each $x^* \in A$, there exists $z^* \in K$ such that $\|x^* - z^*\| \leq \epsilon$. This yields

$$|\langle x^*, x_n \rangle| \leq \epsilon + \sup_{x^* \in K} |\langle x^*, x_n \rangle| \quad (n = 1, 2, \dots).$$

Since X has the DPP_p , it follows from Theorem 3.1 that $\lim_{n \rightarrow \infty} \sup_{x^* \in K} |\langle x^*, x_n \rangle| = 0$. Thus we get $\limsup_n \sup_{x^* \in A} |\langle x^*, x_n \rangle| \leq \epsilon$, which completes the proof (1) \Rightarrow (3).

(1) \Rightarrow (4). Let $(x_n^*)_n$ be a bounded sequence in X^* . Let $\epsilon > 0$ be such that $(x_n^*)_n \subset K + \epsilon B_{X^*}$, $K \subset X^*$ is weakly compact. For each x_n^* , there exists $z_n^* \in K$ such that $\|x_n^* - z_n^*\| \leq \epsilon$. Since K is weakly compact, there exists a weakly convergent subsequence $(z_{k_n}^*)_n$ of $(z_n^*)_n$. By Theorem 5.2, we see that the sequence $(z_{k_n}^*)_n$ is ρ_p^* -convergent and hence $ca_{\mathcal{F}_p}((z_{k_n}^*)_n) = 0$. Note that for any $F \in \mathcal{F}_p$, $F \subset B_X$, we have

$$\begin{aligned} q_F(x_{k_i}^* - x_{k_j}^*) &\leq q_F(x_{k_i}^* - z_{k_i}^*) + q_F(z_{k_i}^* - z_{k_j}^*) + q_F(z_{k_j}^* - x_{k_j}^*) \\ &\leq 2\epsilon + q_F(z_{k_i}^* - z_{k_j}^*), i, j = 1, 2, \dots \end{aligned}$$

This yields

$$ca_{\mathcal{F}_p}((x_{k_n}^*)_n) \leq 2\epsilon + ca_{\mathcal{F}_p}((z_{k_n}^*)_n) = 2\epsilon.$$

Hence, we get $\tilde{ca}_{\mathcal{F}_p}((x_n^*)_n) \leq 2\epsilon$ and then $\tilde{ca}_{\mathcal{F}_p}((x_n^*)_n) \leq 2\omega((x_n^*)_n)$.

(4) \Rightarrow (1). Let $(x_n)_n \in l_p^w(X)$ and let $(x_n^*)_n$ be weakly null in X^* . By (4), we get $\tilde{ca}_{\mathcal{F}_p}((x_n^*)_n) = 0$. A classical diagonal argument yields a subsequence $(x_{k_n}^*)_n$ of $(x_n^*)_n$ which is ρ_p^* -Cauchy. By the completeness of the topology ρ_p^* , we see that the subsequence $(x_{k_n}^*)_n$ is ρ_p^* -convergent. Since $(x_n^*)_n$ is weakly null, $(x_{k_n}^*)_n$ is ρ_p^* -null.

Since $(x_n)_n$ is weakly p -summable, one has

$$|\langle x_{k_n}^*, x_{k_n} \rangle| \leq \sup_i |\langle x_{k_n}^*, x_i \rangle| \rightarrow 0 \quad (n \rightarrow \infty).$$

Then Theorem 3.2 gives (1).

(1) \Rightarrow (5). Let $c > \omega(A)$. Then there exists a weakly compact subset K of X^* such that $\widehat{d}(A, K) < c$. Since X has the DPP_p , it follows from Theorem 3.1 that $\chi_m^p(K) = 0$. Let $\epsilon > 0$ and $L \in \mathcal{F}_p, L \subset B_X$. Then there exists a finite subset $F \subset K$ such that $\widehat{d}(K|_L, F|_L) < \epsilon$, so $\chi(A|_L) \leq c + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\chi(A|_L) \leq c$. By (4.1), we get $\chi_0(A|_L) \leq 2c$. This implies that $\chi_m^p(A) \leq 2c$, which completes the proof. \square

The following quantitative version obviously strengthens the Dunford-Pettis property of order p .

Theorem 5.5. *Let X be a Banach space and $1 < p < \infty$. The following are equivalent:*

- (1) *There is $C > 0$ such that $uc_p^6(T) \leq C \cdot wk_{X^*}(T^*)$ for every operator T from X into any Banach space Y ;*
- (2) *There is $C > 0$ such that $uc_p^6(T) \leq C \cdot wk_{X^*}(T^*)$ for every operator T from X into l_∞ ;*
- (3) *There is $C > 0$ such that $\eta_p(A) \leq C \cdot wk_{X^*}(A)$ for each bounded subset A of X^* ;*
- (4) *There is $C > 0$ such that $uc_p^6(T) \leq C \cdot wk_Y(T)$ for every operator T from X into any Banach space Y ;*
- (5) *There is $C > 0$ such that $uc_p^6(T) \leq C \cdot wk_{l_\infty}(T)$ for every operator T from X into l_∞ .*

Proof. The implication (1) \Rightarrow (2) is trivial with the same constant.

(2) \Rightarrow (3). Assume that there is $C > 0$ such that $uc_p^6(T) \leq C \cdot wk_{X^*}(T^*)$ for every operator T from X into l_∞ . We'll show that (3) holds with the constant $32C$. Let A be a bounded subset of X^* . We may assume that $\eta_p(A) > 0$. Let us fix any $0 < \epsilon < \eta_p(A)$. By the definition of $\eta_p(A)$, there exist a sequence $(x_n^*)_n$ in A and a sequence $(x_n)_n$ in $l_p^w(X)$ with $\|(x_n)_n\|_p^w \leq 1$ such that $|\langle x_n^*, x_n \rangle| > \epsilon$ for each $n \in \mathbb{N}$. Let us define an operator $S : l_1 \rightarrow X^*$ by

$$S((\alpha_n)_n) = \sum_n \alpha_n x_n^*, \quad (\alpha_n)_n \in l_1.$$

As in the proof of Theorem 5.4 in [17], the set $S(B_{l_1})$ is contained in the closed absolutely convex hull of $(x_n^*)_n$ and so $wk_{X^*}(S) \leq 2wk_{X^*}((x_n^*)_n)$. Let $T = S^*J_X : X \rightarrow l_\infty$. By (2) and (5.1), we get $uc_p^6(T) \leq C \cdot wk_{X^*}(T^*)$. Thus

$$\begin{aligned} \epsilon &\leq \limsup_n | \langle x_n^*, x_n \rangle | \leq \limsup_n \|Tx_n\| \\ &\leq uc_p^6(T) \leq C \cdot wk_{X^*}(T^*) \\ &\leq 4C \cdot wk_{l_\infty}(T) \leq 4C \cdot wk_{l_\infty}(S^*) \\ &\leq 16C \cdot wk_{X^*}(S) \leq 32C \cdot wk_{X^*}((x_n^*)_n) \\ &\leq 32C \cdot wk_{X^*}(A) \end{aligned}$$

Since $\epsilon < \eta_p(A)$ is arbitrary, we get the assertion (3).

(3) \Rightarrow (1). Let us suppose that (3) holds with a constant $C > 0$. Let $T \in \mathcal{L}(X, Y)$. Let $(x_n)_n \in l_p^w(X)$ with $\|(x_n)_n\|_p^w \leq 1$. For each $n \in \mathbb{N}$, pick $y_n^* \in B_{Y^*}$ so that $\|Tx_n\| = \langle y_n^*, Tx_n \rangle$. Applying (3) to $A = (T^*y_n^*)_n$, we get

$$\begin{aligned} \limsup_n \|Tx_n\| &= \limsup_n | \langle T^*y_n^*, x_n \rangle | \\ &\leq \limsup_n \sup_{x^* \in A} | \langle x^*, x_n \rangle | \leq \eta_p(A) \\ &\leq C \cdot wk_{X^*}(A) \leq C \cdot wk_{X^*}(T^*), \end{aligned}$$

which yields $uc_p^6(T) \leq C \cdot wk_{X^*}(T^*)$.

Finally, the equivalences of (1) \Leftrightarrow (4) and (2) \Leftrightarrow (5) follow from estimate (5.1). \square

It should be mentioned that the assertion (3) of Theorem 5.5 is a quantitative version of Theorem 3.1.

Definition 5.1. We say that a Banach space X has the *quantitative Dunford-Pettis property of order p* if X satisfies the equivalent conditions of Theorem 5.5.

The following Theorem 5.7 is a quantitative version of Corollary 3.3. To prove it, we need a simple lemma.

Lemma 5.6. *Let X be a closed subspace of a Banach space Y and let A be a bounded subset of X . Then*

$$(5.2) \quad wk_Y(A) \leq wk_X(A) \leq 2wk_Y(A).$$

Proof. We can identify X^{**} with $X^{\perp\perp} \subset Y^{**}$. Under this identification, the *weak*^{*} closure of A in X^{**} is equal to the *weak*^{*} closure of A in Y^{**} . This yields the left inequality immediately. To prove the right inequality of (5.2), let us fix any $c > wk_Y(A)$. Take any $y^{**} \in \overline{A}^{w^*}$. Then there exists $y \in Y$ such that $\|y^{**} - y\| \leq c$. Choose $y^* \in X^\perp$ with $\|y^*\| = 1$ so that $d(y, X) = |\langle y^*, y \rangle|$. Then we get

$$d(y^{**}, X) \leq \|y^{**} - y\| + d(y, X) \leq c + |\langle y^*, y \rangle| = c + |\langle y^*, y^{**} - y \rangle| \leq 2c.$$

Thus $wk_X(A) \leq 2c$. By the arbitrariness of $c > wk_Y(A)$, we obtain $wk_X(A) \leq 2wk_Y(A)$. □

Theorem 5.7. *If X^{**} has the quantitative Dunford-Pettis property of order p , then so is X . More precisely,*

- (a) *If X^{**} satisfies one of the conditions (1), (2), (4) and (5) of Theorem 5.5 with a given constant C , then X satisfies the respective condition of Theorem 5.5 with $16C$;*
- (b) *If X^{**} satisfies the condition (3) of Theorem 5.5 with a given constant C , then X satisfies the respective condition (3) of Theorem 5.5 with C .*

Proof. The assertion (a) follows immediately from the inequality (5.1) and the easy fact that $uc_p^6(T) \leq uc_p^6(T^{**})$ for each operator T . The assertion (b) is a direct consequence of (5.2). □

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